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An analytic solution for weak-field Schwarzschild geodesics

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ABSTRACT

It is well known that the classical gravitational two-body problem can be transformed into a spherical harmonic oscillator by regularization. We find that a modification of the regularization transformation has a similar result to leading order in general relativity. In the resulting harmonic oscillator, the leading-order relativistic perturbation is formally a negative centrifugal force. The net centrifugal force changes sign at 3 Schwarzschild radii, which interestingly mimics the innermost stable circular orbit of the full Schwarzschild problem. Transforming the harmonic-oscillator solution back to spatial coordinates yields, for both time-like and null weak-field Schwarzschild geodesics, a solution for t , r , ϕ in terms of elementary functions of a variable that can be interpreted as a generalized eccentric anomaly. The textbook expressions for relativistic precession and light deflection are easily recovered. We suggest how this solution could be combined with additional perturbations into numerical methods suitable for applications such as relativistic accretion or dynamics of the Galactic Centre stars.

Key words: gravitation – Galaxy: centre.

1 INTRODUCTION

When Schwarzschild geodesics appear in classic tests of general relativity, the important result is an integral over the geodesic: orbital precession or deflection of light. Similarly, in modern tests of relativity involving binary pulsars (for a review, see Will 2006), the observable effects are also cumulative over many orbits.

In the case of the recently discovered S-stars near the Galactic Centre, the cumulative effects of relativity are no longer the principal quantity of interest. The highly eccentric examples from the S-stars (Ghez et al. 2008; Gillessen et al. 2009), which experience a range of gravitational regimes, motivate an interest in tracking relativistic effects as they vary along an orbit. In particular, some recent work has drawn attention to relativistic effects on redshifts near pericentre passage (Zucker et al. 2006; Kannan & Saha 2009; Angéilil & Saha 2010). These effects can be calculated numerically and some of them also by post-Newtonian perturbation theory, but a simpler method is desirable.

Such a method is suggested by the Levi-Civita or Kustaanheimo–Stiefel (LC or KS) regularization, which are transformations of the classical gravitational two-body problem to an equivalent harmonic oscillator. This type of regularization was originally introduced in two dimensions (Levi-Civita 1920) and much later extended to three dimensions (Kustaanheimo 1964; Kustaanheimo & Stiefel 1965). The KS regularization has an extensive literature, including applications to N -body simulations (Aarseth & Zare 1974a,b; Jernigan

& Porter 1989; Mikkola & Aarseth 1993). The classical result suggests that the LC or KS regularization could be used to transform the general relativistic problem into a perturbed harmonic oscillator. We find even better: a modification of the LC or KS transformation acting on the geodesics of the leading-order Schwarzschild metric in the isotropic or harmonic gauge (cf. section 8.2 of Weinberg 1972),

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{2M^2}{r^2} + O\left(\frac{M^3}{r^3}\right) \right) dt^2 + \left(1 + \frac{2M}{r} + O\left(\frac{M^2}{r^2}\right) \right) d\mathbf{x}^2, \quad (1)$$

yields an unperturbed circular/spherical harmonic oscillator. As a result, the solution is analytic. The difference from the classical case is a negative centrifugal-force term in the transformed space. This term encodes the leading-order effects of precession, deflection of light and the innermost stable orbit.

Because orbits in the Schwarzschild space–time do not leave the orbital plane, in this paper we mainly consider the two-dimensional or LC case. The three-dimensional or KS case is similar, but algebraically more complicated, as it involves introducing a fourth spatial dimension.

2 TRANSFORMATION OF THE GEODESIC EQUATIONS

Since regularization is formulated in the language of Hamiltonians, we begin by expressing the geodesic equations in the Hamiltonian

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form. A convenient expression for the Hamiltonian is (e.g. equation 25.10 of Misner, Thorne & Wheeler 1973)

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \quad (2)$$

Considering metric (1), we have

$$H = -\frac{1}{2} \left(1 + \frac{2M}{r} + \frac{2M^2}{r^2} \right) p_t^2 + \left(1 - \frac{2M}{r} \right) \frac{p^2}{2}. \quad (3)$$

This retains the leading-order Newtonian corrections, which are $O(M/r)$ in the spatial part and $O(M^2/r^2)$ in the temporal part, and neglects higher orders. Note that the Hamiltonian (3) has 4 degrees of freedom: time t being a coordinate and p_t its conjugate momentum, while the affine parameter λ is the independent variable. We assume units with $G/c^2 = 1$.

The Hamiltonian being independent of t , it follows that p_t is a constant and Hamilton's equation for t reads as

$$\frac{dH}{dp_t} = \frac{dt}{d\lambda} = - \left(1 + \frac{2M}{r} + \frac{2M^2}{r^2} \right) p_t, \quad (4)$$

where λ denotes the affine parameter. On choosing $p_t = -1$ (which we are free to do, as this amounts to choosing units for λ) and discarding a constant, we arrive at the Hamiltonian

$$H = -\frac{1}{2} \left(\frac{2M}{r} + \frac{2M^2}{r^2} \right) + \left(1 - \frac{2M}{r} \right) \frac{p^2}{2}. \quad (5)$$

As is well known from the spherical symmetry of the Schwarzschild space-time, geodesics are confined to a plane. Without loss of generality we can choose a two-dimensional coordinate system in the x, y plane. (Below, at the end of this section, we briefly indicate the procedure without this simplification.)

We now apply a regularization transformation. In the x, y plane, we introduce two new coordinates, given by the real and imaginary parts of the complex number,

$$Q = \sqrt{x + iy}. \quad (6)$$

Hence, $r = |Q|^2$. The conjugate momentum components are given by the real and imaginary parts of a complex number P , which satisfies

$$p_x + ip_y = \frac{Q^* P}{2|Q|^2}. \quad (7)$$

On multiplying the above equation by its complex conjugate, we arrive at the transformation

$$p^2 = \frac{|P|^2}{4|Q|^2}. \quad (8)$$

The P, Q are known as LC variables. A proof that they are indeed canonical appears in several sources (e.g. Saha 2009), and we do not repeat it here. Transforming (5) and rearranging yield the Hamiltonian in the LC variables

$$H = -\frac{1}{2} \left(\frac{2M}{|Q|^2} + \frac{2M^2}{|Q|^4} \right) + \left(\frac{1}{|Q|^2} - \frac{2M}{|Q|^4} \right) \frac{|P|^2}{8}. \quad (9)$$

To complete the regularization, we invoke a Poincaré time transformation which involves introducing a scaled time variable related to the affine parameter by

$$d\lambda = g(P, Q) ds, \quad (10)$$

where $g(P, Q)$ can be any function, and define a new Hamiltonian

$$\Gamma \equiv g(H - E). \quad (11)$$

The Γ Hamiltonian (11) preserves the Hamiltonian form of the equations of motion, provided the constant E is the initial value of the original Hamiltonian. We choose

$$g = \left(\frac{1}{|Q|^2} - \frac{2M}{|Q|^4} \right)^{-1} = \frac{|Q|^2}{1 - 2M/|Q|^2}, \quad (12)$$

which approximates to $|Q|^2 + 2M$ for large $|Q|$. The time-transformed Hamiltonian takes the form

$$\Gamma = \frac{|P|^2}{8} - E|Q|^2 - 3\frac{M^2}{|Q|^2} - M(1 + 2E) + O(|Q|^{-4}). \quad (13)$$

We remark that the classical case uses $g = |Q|^2$, and only the first two terms in (13) are present in the Γ Hamiltonian, up to a constant. However, as is evident from (13), the weak-field Schwarzschild case also yields a harmonic oscillator under a regularization transformation. The essential difference is in the extra $1/|Q|^2$ term which encodes the leading-order effects of general relativity by altering the classical angular momentum in (13). Thus, the key modification from the classical case which allows the formulation of an analytically solvable, relativistic Hamiltonian is the choice of $g(P, Q)$ in equation (12).

If we do not restrict the coordinates to the plane, the LC transformation must be replaced with a KS transformation. Although the transformation itself is far more complicated (involving four spatial dimensions) the time transformation and Γ Hamiltonian (13) remain the same, except that $|P|$ and $|Q|$ are lengths in four Euclidean dimensions. This is easily seen on comparing the above with the KS regularization of the Kepler problem (see e.g. section 5 of Saha 2009).

3 SOLUTIONS

Continuing in two dimensions, it is possible to put the Γ Hamiltonian in a more recognizable form by transforming to polar coordinates (Q_r, Q_ϕ, P_r, P_ϕ):

$$\Gamma = \frac{P_r^2}{8} + \frac{(P_\phi^2 - 6(2M)^2)}{8Q_r^2} - EQ_r^2 - M(1 + 2E) + O(Q_r^{-4}), \quad (14)$$

which for negative values of E is the Hamiltonian for a circular classical harmonic oscillator with squared angular momentum decreased by $6(2M)^2$ from the equivalent Kepler problem.

We remark that the Γ Hamiltonian appears to depend only on terms up to $1/r$, and it would seem that by leaving out terms of order $|Q|^{-4}$ we have omitted relativistic effects. However, this is not the case, since the time equation now hides a factor of r . Thus, one should read terms of order $|Q|^{-n}$ as terms of order $r^{-(n/2+1)}$ for $n = 0, 1, 2, \dots$. This provides some insight into why the Hamiltonian now has soluble equations of motion: we have pushed a factor of r into the time equation.

We introduce the constants

$$M' \equiv M(1 + 2E) \quad \text{and} \quad P_\phi'^2 \equiv P_\phi^2 - 6(2M)^2. \quad (15)$$

Dropping higher order terms, the Hamiltonian becomes

$$\Gamma = \frac{P_r^2}{8} + \frac{P_\phi'^2}{8Q_r^2} - EQ_r^2 - M', \quad (16)$$

which is identical to the transformed Hamiltonian of a particle with angular momentum P_ϕ' in a central force potential of the form $V(r) = M'/r$ or equivalently to the Kepler problem where M is replaced by M' and P_ϕ^2 is replaced by $P_\phi'^2$.

Hamilton's equations of motion are

$$\begin{aligned}\frac{dQ_r}{ds} &= \frac{P_r}{4} \\ \frac{dP_r}{ds} &= 2EQ_r + \frac{P_\phi^2}{4} \frac{1}{Q_r^3} \\ \frac{dQ_\phi}{ds} &= \frac{P_\phi}{4} \frac{1}{Q_r^2},\end{aligned}\quad (17)$$

where P_ϕ is a constant. On combining with the time transformation (10), equation (4) for the time coordinate becomes

$$\begin{aligned}dt &= \left(Q_r^2 + 2M + \frac{2M^2}{Q_r^2} \right) \frac{ds}{1 - 2M/Q_r^2} \\ &\simeq \left[4M + Q_r^2 + 6\frac{M^2}{Q_r^2} + O(Q_r^{-4}) \right] ds,\end{aligned}\quad (18)$$

where we have again used the approximation $Q_r^2 \gg 2M$.

We remark that $P_\phi = 2p_\phi$, which is the weak-field relationship between the LC angular momentum and angular momentum in standard coordinates. This can be seen from a comparison of the third of equations (17) and the corresponding Hamilton equation applied to (5) in the time variable s .

3.1 Bound orbits

For negative values of E , the solutions are bound orbits. We define a 'classical' and a 'relativistic' semimajor axis

$$a \equiv \frac{M}{2|E|}, \quad a' \equiv \frac{M'}{2|E|}, \quad (19)$$

the eccentric anomaly

$$\beta = \sqrt{2|E|}s \quad (20)$$

and the eccentricity

$$e = \sqrt{1 + \frac{P_\phi^2 E}{2M^2}} \quad (21)$$

in the standard way. Solving equation (18) gives an implicit equation for the coordinate time t in terms of β and solving equations (17) via quadrature gives the following solutions for the LC variables in terms of elementary functions of β :

$$\begin{aligned}Q_r^2(\beta) &= a' [1 - e \cos \beta] \\ Q_\phi(\beta) &= \frac{P_\phi}{P'_\phi} \tan^{-1} \left[\sqrt{\frac{1+e}{1-e}} \tan \left(\frac{\beta}{2} \right) \right] \\ t &= \sqrt{\frac{a'^3}{M'}} [\beta - e \sin \beta] + \frac{4M}{\sqrt{2|E|}} \beta + \frac{6(2M)^2}{P_\phi} Q_\phi,\end{aligned}\quad (22)$$

where we have chosen $Q_\phi(0) = 0$ and $Q_r^2(0) = a'(1 - e) = r_{\min}$. The LC radial momentum $P_r(\beta)$ is then generated from the first of equations (17) and, with the above choice of $Q_r^2(0)$, the initial radial momentum vanishes.

Note that in terms of the classical semimajor axis of the Kepler problem, the quantity $a' = a - M$. So it is simple to show that equations (22) reduce to the Kepler LC equations of motion when M is small.

3.2 Unbound trajectories

For the unbound case the Hamiltonian and thus the equations of motion only change by the sign of E which appears in a', e and β .

With this substitution, the unbound solutions become

$$\begin{aligned}Q_r^2(\beta) &= a' [e \cosh \beta - 1] \\ Q_\phi(\beta) &= \frac{P_\phi}{P'_\phi} \tan^{-1} \left[\sqrt{\frac{e+1}{e-1}} \tanh \left(\frac{\beta}{2} \right) \right] \\ t &= \sqrt{\frac{a'^3}{M'}} [e \sinh \beta - \beta] + \frac{4M}{\sqrt{2|E|}} \beta + \frac{6(2M)^2}{P_\phi} Q_\phi,\end{aligned}\quad (23)$$

where we have chosen $Q_r^2(0) = a'(e - 1)$, which in the unbound case is the point of closest approach.

3.3 Light rays

Null geodesics are the solutions to Hamilton's equations when H in equation (3) is set to zero. Since we discarded a constant of $-1/2$ in the derivation of the LC Hamiltonian, the null solutions can be found by assigning E in the unbound case the value of $1/2$. This amounts to redefining the constants a', β and e in equations (23) such that the null equations of motion become

$$\begin{aligned}Q_r^2(s) &= 2M [e_n \cosh s - 1] \\ Q_\phi(s) &= \frac{P_\phi}{P'_\phi} \tan^{-1} \left[\sqrt{\frac{e_n+1}{e_n-1}} \tanh \left(\frac{s}{2} \right) \right] \\ t &= 2M [e_n \sinh s - s] + 4Ms + \frac{6(2M)^2}{P_\phi} Q_\phi,\end{aligned}\quad (24)$$

with

$$e_n = \sqrt{1 + \frac{P_\phi^2}{16M^2}} \quad (25)$$

and $Q_r^2(0) = 2M(e_n - 1)$.

4 PROPERTIES OF THE SOLUTION

Now that we have derived the bound, unbound and null equations of motion we show that, to first order, they reproduce the predictions for geodesics in a Schwarzschild space-time, namely those of orbital precession, the deflection of light and the innermost stable circular orbit (ISCO).

4.1 Orbital precession

The pre-factor of the middle line in (22) automatically gives the precession rate of orbits and tells us that the precession is due to the $-6(2M)^2$ perturbation of the classical squared angular momentum. This is equivalent to the conventional interpretation of precession being caused by an additional centrifugal-force term. Substituting for P'_ϕ , we have

$$\frac{P_\phi}{P'_\phi} = \left[1 - 6 \frac{(2M)^2}{P_\phi^2} \right]^{-1/2}. \quad (26)$$

Since $P_\phi^2 \propto Q_r^4$, then $P_\phi^2 \gg (2M)^2$ and we may write

$$\left[1 - 6 \frac{(2M)^2}{P_\phi^2} \right]^{-1/2} \simeq 1 + 3 \frac{(2M)^2}{P_\phi^2}. \quad (27)$$

We note that, due to the complex square-root nature of the LC transformation, $Q_\phi = \frac{1}{2}\phi$. Thus, as the solution evolves through one period Q_ϕ will increase by

$$\pi + 3\pi \frac{(2M)^2}{P_\phi^2} \text{ rad}, \quad (28)$$

giving an orbital precession rate of

$$\Delta Q_\phi = 3\pi \frac{(2M)^2}{P_\phi^2} \frac{\text{rad}}{\text{orbit}}. \quad (29)$$

Converting back to non-LC coordinates and expressing the previous equation in terms of the semilatus rectum of the orbit $\alpha = a(1 - e)$, where a is the classical semimajor axis, the precession rate becomes

$$\Delta\phi = 6\pi \frac{MG}{c^2\alpha} \frac{\text{rad}}{\text{orbit}}, \quad (30)$$

as in the conventional treatment (e.g. Weinberg 1972).

4.2 Deflection of light

We may similarly derive the deflection angle of a light ray. Imagine that the light ray starts infinitely far from the Schwarzschild mass at $t = -\infty$, approaches the point of closest approach at $t = 0$ and continues on to $t = \infty$. If there is no deflection, Q_ϕ will sweep out $\pi/2$ rad since $Q_\phi = \frac{1}{2}\phi$. If there is deflection, the total difference in $Q_\phi(-\infty)$ and $Q_\phi(\infty)$ will be greater than $\pi/2$. To determine the deflection angle, we compute

$$\begin{aligned} \Delta Q_\phi &= 2[Q_\phi(\infty) - Q_\phi(0)] - \frac{\pi}{2} \\ &= 2 \frac{P_\phi}{P'_\phi} \tan^{-1} \left[\frac{e_n + 1}{\sqrt{e_n^2 - 1}} \right] - 0 - \frac{\pi}{2}, \end{aligned} \quad (31)$$

where Q_ϕ is the null solution for Q_ϕ . Substituting for P'_ϕ and the expression for e_n in equation (25), invoking the binomial approximation for $P_\phi^2 \gg M^2$ and $P_\phi^2 \gg M^2$ and neglecting terms of order M^2 and greater, we find

$$\Delta Q_\phi = 2(1 + O(M^2)) \tan^{-1} \left[1 + \frac{4M}{P'_\phi} + O(M^2) \right] + 0 - \frac{\pi}{2}. \quad (32)$$

Using the expansion

$$\tan^{-1}(1 + x) = \frac{\pi}{4} + \frac{1}{2}x + O(x^2), \quad (33)$$

we obtain

$$\Delta Q_\phi = \frac{\pi}{2} + 2 \left[\frac{2M}{P'_\phi} - O(M^2) \right] - \frac{\pi}{2}. \quad (34)$$

Converting back to non-LC coordinates, we gain a factor of $1/2$ since the LC angular momentum P'_ϕ is twice the non-LC angular momentum P_ϕ . We may treat P'_ϕ as the corrected (relativistic) angular momentum, which is equal to the impact parameter b for photons (with the speed of light set to unity). Thus we arrive at the expression, to first order in M , for the angle by which light is deflected due to a spherically symmetric mass distribution:

$$\Delta\phi_{\text{light}} = \frac{4MG}{b} \simeq \frac{4MG}{r_0}, \quad (35)$$

where to this order the impact parameter can be replaced by r_0 , the point of closest approach.

4.3 Innermost stable orbits

Due to the weak-field approximation used to derive the geodesic equations of motion, the solutions do not exhibit an event horizon. Remarkably though, the solutions do reproduce the phenomenon of an ISCO. This is due to the centrifugal term $6(2M)^2/Q_r^2$ in the LC Hamiltonian (14).

From the first two of Hamilton's equations (17) for a bound orbit, we find

$$\frac{d^2 Q_r}{ds^2} = -\frac{|E|}{2} Q_r \left[1 - \frac{P_\phi^2}{8|E|Q_r^4} \right] \quad (36)$$

and note that equation (36) becomes a one-dimensional harmonic oscillator in the radial coordinate (i.e. the condition for radial free fall or equivalently $e \rightarrow 1$) when $P_\phi^2 = 0$. Writing out P'_ϕ , the condition becomes

$$P_\phi^2 = 6(2M)^2. \quad (37)$$

Dividing by $|E|$ and using the definitions of the classical semimajor (a) and semiminor (b) axes of the orbit, this condition becomes

$$b^2 = 6aM, \quad (38)$$

which in the case of a circular orbit ($a = b$) is precisely the ISCO predicted by the conventional analysis of time-like geodesics in a Schwarzschild metric. Note that although this relation has been derived for the classical semimajor axis, one can convert to the standard radial coordinate of the Schwarzschild metric and find an identical result as follows. First, convert to the relativistic semimajor axis, $a' = a - M$ which, for the above ISCO, gives $a' = 5M$. Then recall that the standard Schwarzschild radial coordinate R is given in terms of the isotropic radial coordinate by (see e.g. section 8.2 of Weinberg 1972)

$$R = r \left(1 + \frac{M}{2r} \right)^2 \simeq r \left(1 + \frac{M}{r} \right), \quad (39)$$

where the approximation is made to be consistent with the derivation of the equations of motion. For $a' = r = 5M$, this gives $R = 6M$ as expected.

To interpret the ISCO derived here, we observe from equation (36) that the condition for zero radial acceleration is

$$Q_r^2 = \sqrt{\frac{P_\phi^2 - 6(2M)^2}{8|E|}}. \quad (40)$$

Now consider a classical orbit with a fixed P_ϕ^2 and imagine 'turning on' relativity adiabatically keeping P_ϕ^2 constant. As relativity is turned on, the negative centrifugal-force term increases and effectively reduces P_ϕ^2 . If $P_\phi^2 > 6(2M)^2$ to begin with, then the classical orbit will shrink by the appropriate amount in the presence of relativistic effects. But if we originally had $P_\phi^2 \leq 6(2M)^2$, then as relativity is turned on equation (40) shows that Q_r^2 would shrink to 0 implying that all orbits for which $P_\phi^2 \leq 6(2M)^2$ are unstable.

Also note that the standard coordinate singularity at $R = 2M$ is mapped to $r = M$ in the approximate isotropic coordinates. This yields the following interpretation for the LC equations of motion. Substituting $a' = a - M$ in the first of equations (22) for the LC radial coordinate and restricting to the circular case where $e = 0$, we find

$$Q_r^2(\beta) + M = a, \quad (41)$$

which has the interpretation that the LC radial coordinate is measured not from the origin of the corresponding Kepler problem but from the event horizon predicted by the Schwarzschild metric. This does not carry the interpretation of a horizon since when $e \neq 0$ orbits can still come arbitrarily close to $r = 0$ and continue outside of $r = M$. However, it does add an interesting interpretation to the regularization transformation. For circular orbits, the regularization essentially cuts out the area inside the event horizon and stitches it back together mapping a circle to the origin. For elliptical orbits, one can imagine an analogous interpretation.

Since these equations of motion predict the correct ISCO and exhibit special behaviour at the Schwarzschild event horizon, perhaps they could be particularly useful for simple approximate modelling of relativistic accretion discs.

4.4 Error terms

Working backwards from the approximate Hamiltonian in the time variable s (equation 13) by solving equation (11) for H , we can recover the error terms in our approximate Hamiltonian. Solving

$$\Gamma = \frac{|Q|^2}{1 - 2M/|Q|^2}(H - E) \quad (42)$$

with no further approximations and converting to non-LC coordinates yield what we may call a surrogate Hamiltonian:

$$H_{\text{surr}} = -\frac{1}{2} \left[1 + \frac{2M}{r} + \frac{2M^2}{r^2} - \left(\frac{12M^3}{r^3} \right) \right] p_t^2 + \left(1 - \frac{2M}{r} \right) \frac{p^2}{2} + \frac{4M^2}{r^2} E \quad (43)$$

for which the analytic solution is exact. For time-like geodesics E is small and for null geodesics $1/r^2$ is a second-order correction. Thus, to obtain a first-order expression for error in the metric components we may neglect the term containing E .

If it is bothersome that the Hamiltonian in equation (43) is dependent on E , one can eliminate it by a further modification. Let us add an $E/|Q|^4$ term to the Γ Hamiltonian:

$$\Gamma' = \Gamma + \frac{4M^2}{|Q|^4} E, \quad (44)$$

which makes only a higher order change to the solutions. Introducing H'_{surr} by

$$\Gamma' = |Q|^2(1 + 2M/|Q|^2 + 4M^2/|Q|^4)(H'_{\text{surr}} - E), \quad (45)$$

one finds

$$H'_{\text{surr}} = -\frac{1}{2} \left[\frac{1 + 4M/r + 10M^2/r^2}{1 + 2M/r + 4M^2/r^2} \right] p_t^2 + \left[\frac{1}{1 + 2M/r + 4M^2/r^2} \right] \frac{p^2}{2}, \quad (46)$$

which, when expanded in $1/r$ to the appropriate order, is identical to (43) less the term dependent on E . The Hamiltonian (46) is then the Hamiltonian for which equations (22), (23) and (24) are exact solutions.

5 DISCUSSION

We have derived time-like and null geodesics in the leading-order Schwarzschild metric in terms of elementary functions. Expressions (22) for bound orbits and (23) for unbound orbits, together with (24) for light rays, are all simple generalizations of well-known expressions in classical celestial mechanics. The usual formulae for relativistic orbital precession and light deflection are easily recovered. A feature resembling the ISCO in the full Schwarzschild metric is also present.

The technique we have used is a modification of the LC or KS regularization transformation and transforms the geodesic equation into a spherical harmonic oscillator. The simplicity of the result, notwithstanding the non-trivial route used to derive it, hints at some underlying symmetry in the Schwarzschild problem. We speculate

that it is somehow related to the separability of the Hamilton–Jacobi and other equations in the Schwarzschild and Kerr metrics (cf. Chandrasekhar 1983) but have not attempted to investigate this.

As mentioned in Section 1, the original motivation for this work was to find useful formulae applicable to the highly eccentric Galactic Centre stars, whose orbits pass through a large range of gravitational regimes. Future observations of these stars aiming to detect relativistic effects will require computation of relativistic effects on both stellar orbits and light rays at many points along an orbit, for many orbits, in order to fit the orbital parameters. The solutions in this paper allow a simpler, more efficient method for carrying out those computations. The analytic solutions will not be sufficient on their own because the Galactic Centre stars also experience additional Newtonian perturbations due to local matter (Mikkola & Merritt 2008), but they can be incorporated into numerical methods, specifically generalized leapfrog integrators. Such algorithms evolve alternately under two Hamiltonians, which are integrable separately. The idea goes back to Wisdom & Holman (1991) and Kinoshita, Yoshida & Nakai (1991). Some recent developments on adaptive stepsizes appear in Emel’yanenko (2007) and are applied to the specific problem of Galactic Centre stars in Preto & Saha (2009). We note, however, that this work is limited to test particles and hence will not be applicable for binary orbits or self-gravitating disc simulations unless a generalization is found.

Another potential application may be the use of the solutions in relativistic disc simulations as an alternative to the widely used pseudo-Newtonian potentials (see especially Paczyński & Wiita 1980; Artemova, Bjoernsson & Novikov 1996; Abramowicz 2009), an advantage being that the solutions in this paper are well-defined approximations and include a more complete repertoire of general-relativistic effects for the same computational budget.

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